Topics in Number Theory 2023
Homework 1 – Submit by 21-02-2023, 13:00
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Exercise 1: (1+2+2 points)
Let p be an odd prime, let $L = \mathbb{Q}(\zeta_p)$, where ζ_p is a primitive pth root of unity, with minimal polynomial Φ_p.
(a) Compute $\Phi'_p(\zeta_p)$.
(b) Compute $N(1-\zeta_p)$.
(c) Compute $\operatorname{disc}(1, \zeta_p, \dots, \zeta_p^{p-2})$.

Exercise 2: (3 points)
Let $L = \mathbb{Q}(\sqrt{d})$, with $d \in \mathbb{Z}$ squarefree. Show that $\mathbb{Z}_L = \mathbb{Z}[\alpha]$, where

$$\alpha = \begin{cases} \sqrt{d}, & \text{if } d \equiv 2,3 \pmod 4 \\ \frac{1+\sqrt{d}}{2}, & \text{if } d \equiv 1 \pmod 4 \end{cases}.$$

Exercise 3: (2+2+1+4 points)
Let $L = \mathbb{Q}(\sqrt{2}, \sqrt{3})$ and let $\beta = a + b\sqrt{2} + c\sqrt{3} + d\sqrt{6} \in L$ for some $a, b, c, d \in \mathbb{Q}$.
(a) Find a primitive element of L.
(b) Show that β is integral over $\mathbb{Z}$ if and only if for all $m \in \{2, 3, 6\}$ we have:

$$\mathrm{Tr}_{L/\mathbb{Q}(\sqrt{m})}(\beta) \in \mathbb{Z}_{\mathbb{Q}(\sqrt{m})} \quad \text{and} \quad \mathrm{N}_{L/\mathbb{Q}(\sqrt{m})}(\beta) \in \mathbb{Z}_{\mathbb{Q}(\sqrt{m})}$$

(c) Suppose that $\beta \in \mathbb{Z}_L$. Show that $2a, 2b, 2c, 2d \in \mathbb{Z}$.
(d) Show that

$$\left(1, \sqrt{2}, \sqrt{3}, \frac{\sqrt{2}+\sqrt{6}}{2}\right)$$

is an integral basis of L.

Exercise 4: (optional)
Let $\beta = \cos(72°) \in \mathbb{R}$ and let ζ_5 be the primitive 5th root of unity $\beta + i\sin(72°)$. Let $L = \mathbb{Q}(\zeta_5)$ and let $\alpha = a + b\zeta_5 + c\zeta_5^2 + d\zeta_5^3 \in L$, where $a, b, c, d \in \mathbb{Q}$.
(a) Compute $\mathrm{Tr}(\alpha)$.
(b) Show that

$$16\beta^5 - 20\beta^3 + 5\beta = 1.$$

(c) Deduce that

$$\beta = \frac{\sqrt{5}-1}{4}.$$

(d) Show that $\sqrt{5} \in L$. *(Hint: Consider $\zeta_5 + \zeta_5^4$.)*
(e) Compute $\mathrm{N}_{L/\mathbb{Q}(\sqrt{5})}(\alpha)$.
(f) Use (e) to compute $\mathrm{N}_{L/\mathbb{Q}}(\alpha)$.